Maximizing a Monotone Submodular Function Subject to a Covering and a Packing Constraint

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Highlights of Algorithms

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Joint work with ***Sumedha Uniyal***

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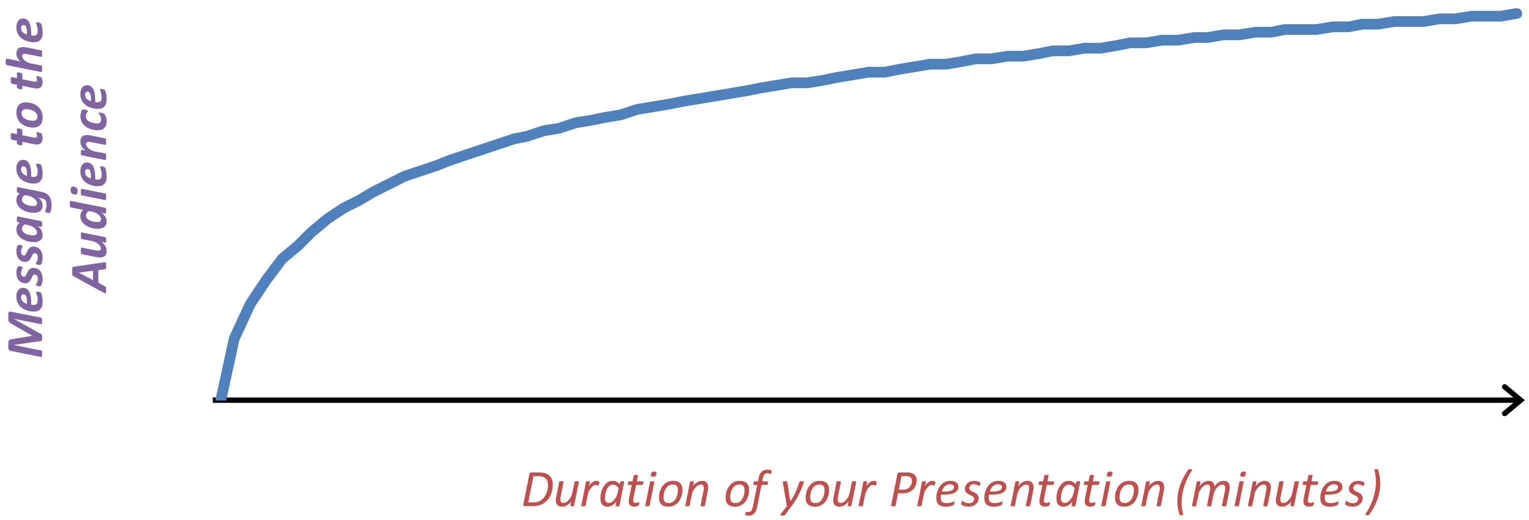
Submodular Functions



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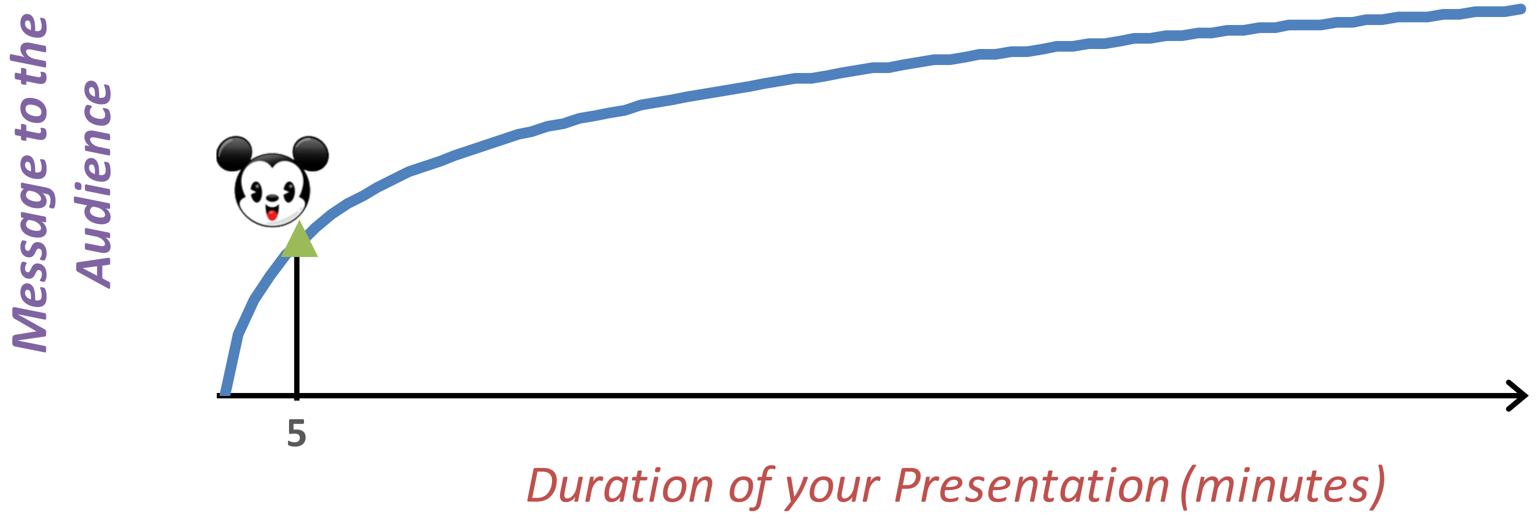

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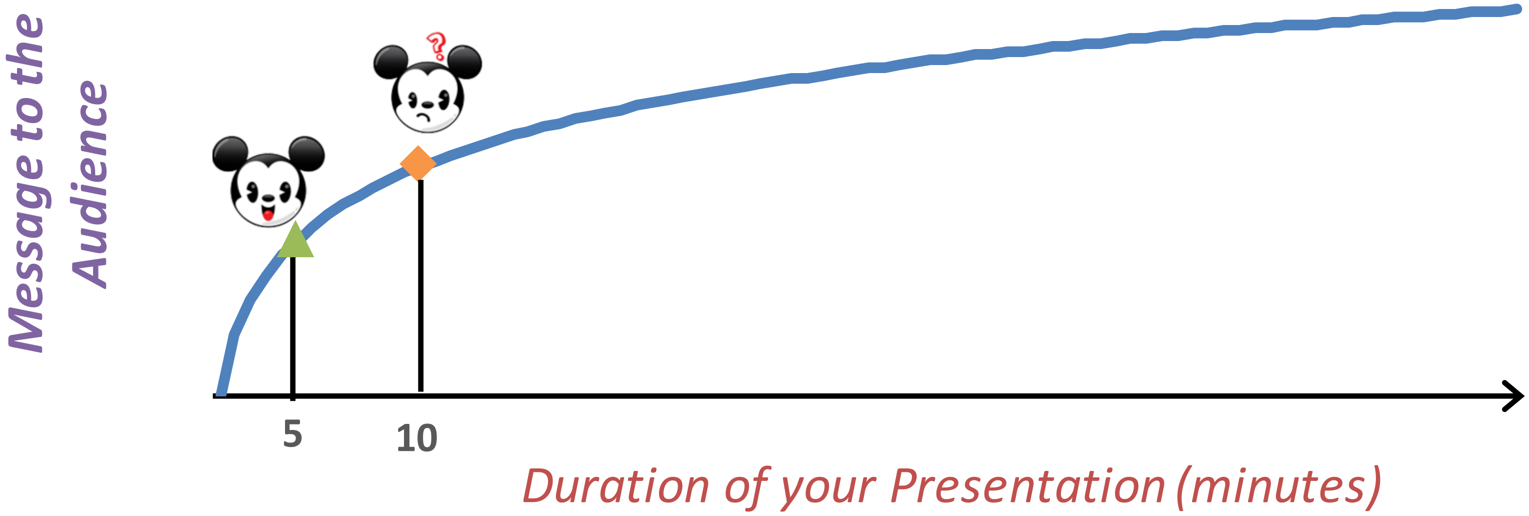

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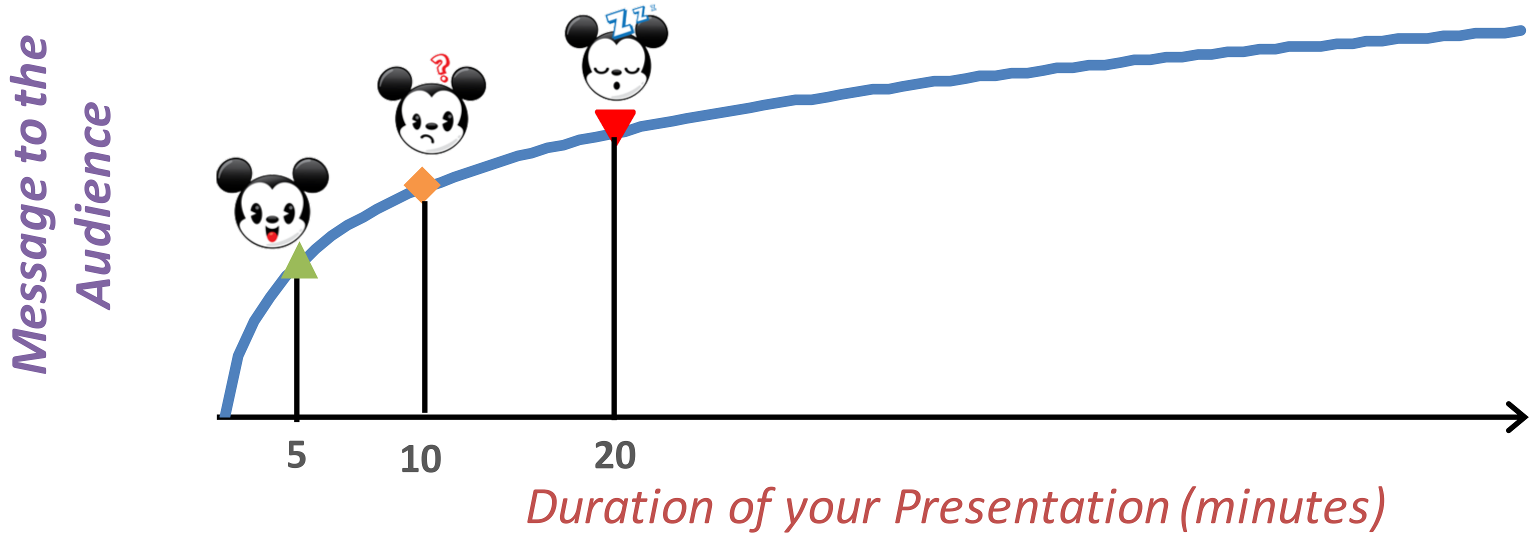

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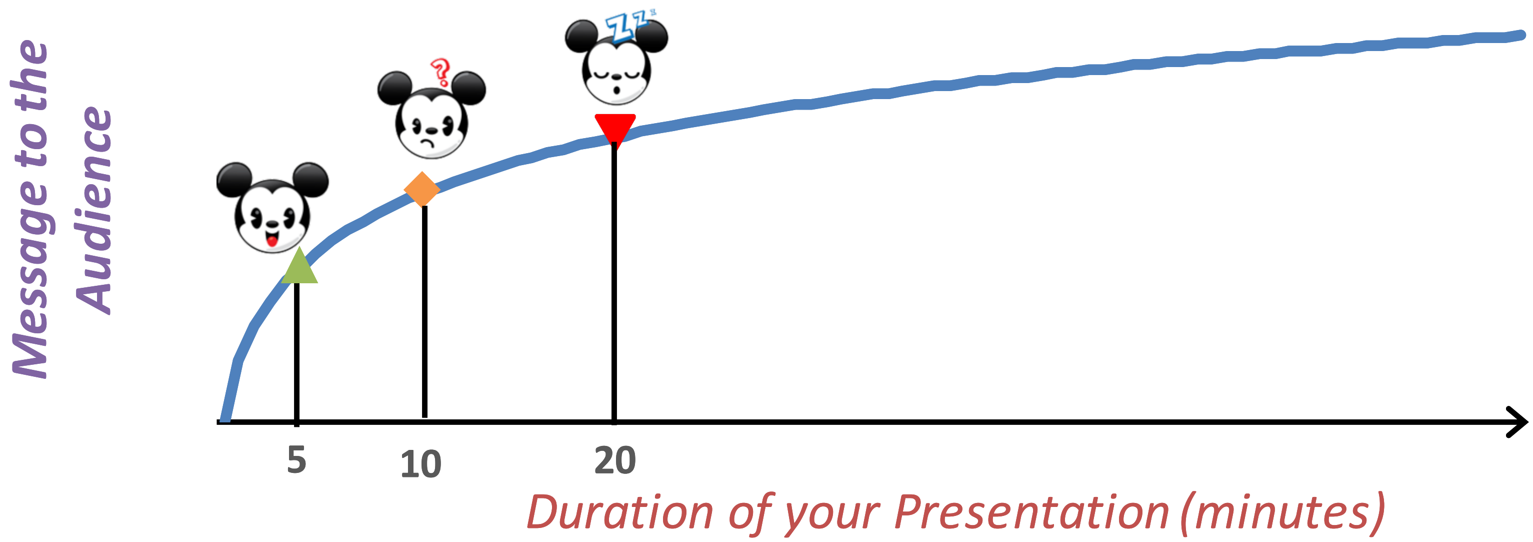
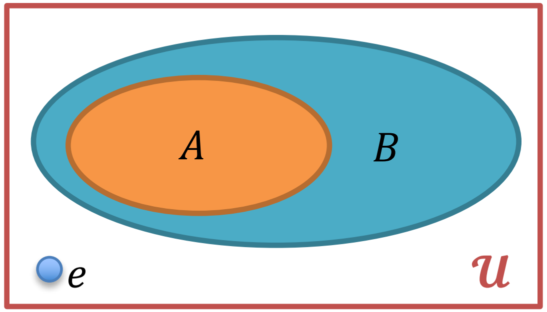


Captures many combinatorial optimization problems like

- *Max-k-Coverage*
- *Max-Cut*
- *Max Facility Location …*

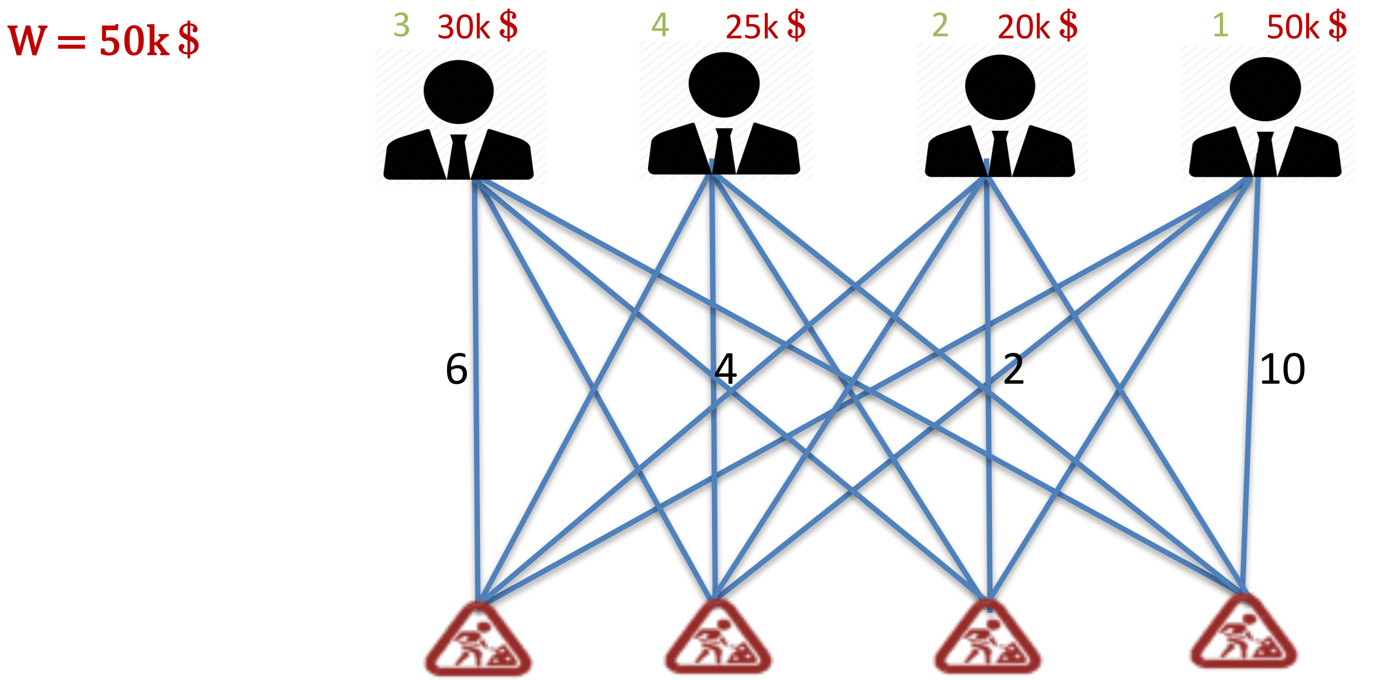


A set function on all subsets of $\mathcal{U}$, $f: 2^{|\mathcal{U}|} \rightarrow \mathbb{R}^+$ such that $\forall A \subseteq B \subseteq \mathcal{U}, e \in \mathcal{U} \backslash B$,

$$f(A \cup \{e\}) - f(A) \geq f(B \cup \{e\}) - f(B) \text{ (Submodular)}$$

$$f(B) \geq f(A) \text{ (Monotone)}$$

Covering and Packing Constraints



For each element $e \in \mathcal{U}$, we are give some weight a profit function $p: \mathcal{U} \to \mathbb{N}$, a weight function $w: \mathcal{U} \to \mathbb{N}$, a profit requirement P and a budget B.

Goal: max $\boldsymbol{f(A)}$ for some $A \subseteq \mathcal{U}$ such that

$$p(A) \geq P \text{ and } w(A) \leq W$$

Feasibility NP-Hard (Subset-Sum Problem)

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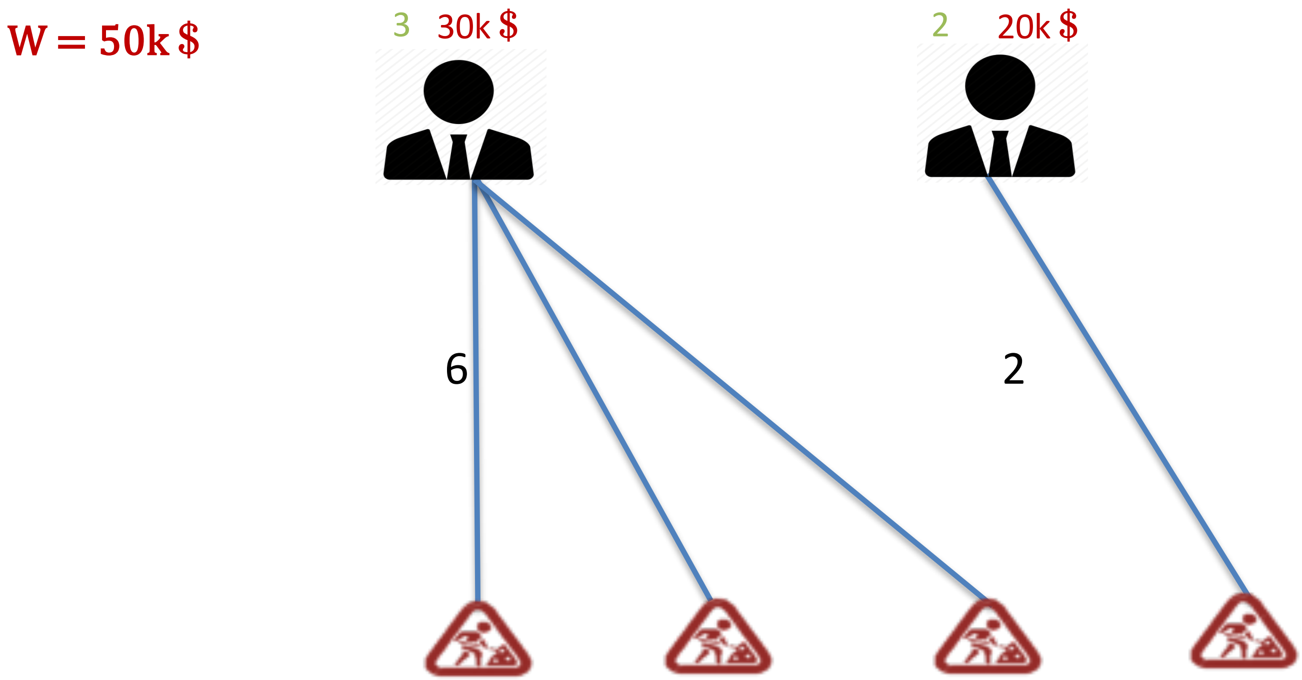

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State of the Art Results

- ***Monotone Submodular Maximization***
  - ***Under Cardinality Constraint***
    - Greedy 1-1/e – apx [Nemhauser et al. , Math. Program.'78]
    - Hardness: 1-1/e [Nemhauser et al. , Math. O.R.., 1978], [Feige, STOC'96]
  - ***Under Packing Constraint***
    - Greedy 1-1/e – apx [Sviridenko, O.R. Lett.'04]
  - ***O(1) Packing Constraints***
    - *Multilinear relaxation: 1-1/e – ε – apx* [Kulik et al. , SODA'09]
  - ***Matroid Constraints***
    - *Multilinear relaxation: 0.309 – apx* [Vondrák, STOC'08], [Calinescu et al., IPCO'07]

- ***Capacitated k – Median Problem***
  - Hardness: 1.736 (follows from the uncapacitated case)
  - Approximation factor: O(log n) [Folklore]
  - O(1) approximation algorithms under capacity/cardinality violations.

Algorithms

- A greedy based dynamic program.
- Outputs an *approximate solution* which is best of all possible greedy chains.

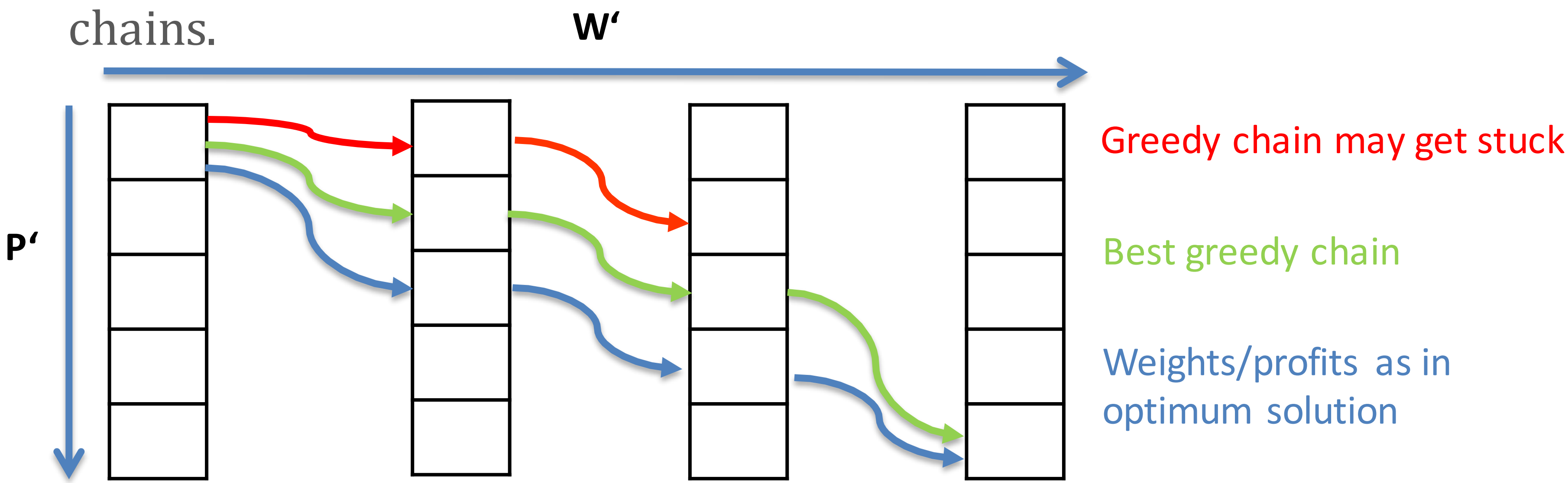

Table entry: $T[l, P', W']$ stores l-element approximate solution with profit P' and weight W'.
To compute $T[l, P', W']$ pick best way to extend an entry $T[l-1, P'', W'']$ by one element in a feasible manner.

Output: Best entry $T\ [l, P', W']$ with $P' \geq \frac{P}{2}$ and $W' \leq W$.

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Forbidden Sets

Crucial to reduce the profit violation from $2+\varepsilon$ down to $1+\varepsilon$.

Works for one covering and one packing constraint.

Idea: For each $T\ [P', W']$, forbid the suffix set $F_{W'}$ of ordered set by non-increasing $\frac{p(e)}{w(e)}$ such that

$p(T\ [P', W'] \cup F_{W'}) \geq P$ and $w(T\ [P', W'] \cup F_{W'}) \leq W$.

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Main Theorems

Theorem 1. *There is an algorithm for maximizing a monotone submodular function subject to* ***O(1)*** *covering and one packing constraint that outputs for any* $\varepsilon > 0$ *in* $n^{O(1/\varepsilon)}$ *time a* ***4****-approximate solution with profit at least* $\left(\frac{1}{2} - \varepsilon\right) P$ *and with weight at most* $(1+\varepsilon)W$.

Theorem 2. *There is an algorithm for maximizing a monotone submodular function subject to one covering and one packing constraint that outputs for any* $\varepsilon > 0$ *in* $n^{O(1/\varepsilon)}$ *time a* ***4****-approximate solution with profit at least* $(1-\varepsilon)P$ *and with weight at most* $(1+\varepsilon)W$.

Corollary. *There is a* ***2.6****- approximation algorithm for k-median problem with non-uniform and hard capacities if the underlying metric space has only two possible distances.*

Factor-revealing LP

min a_n *subject to* **(LP)**

$$a_1 \geq \left(1 - \frac{i}{n}\right) o_1;$$

$$a_i \geq a_{i-1} + \left(1 - \frac{i}{n}\right) o_i \quad \forall i \in [n] \backslash \{1\};$$

$$a_i \geq \frac{i}{n}\left(1 - \sum_{j=1}^{i} o_j\right) \quad \forall i \in [n];$$

$$a_i \geq 0, \; o_i \geq 0 \quad \forall i \in [n].$$

max $\sum_{i=1}^{n} \frac{i}{n} y_i$ *subject to* **(DUAL)**

$$x_i + y_i - x_{i+1} \leq 0 \quad \forall i \in [n-1];$$

$$x_n + y_n \leq 1;$$

$$\sum_{j=i}^{n} \frac{j}{n} y_j - \left(1 - \frac{i}{n}\right) x_i \leq 0 \quad \forall i \in [n];$$

$$x_i \geq 0, \; y_i \geq 0 \quad \forall i \in [n].$$

Theorem 3. *There is an algorithm for maximizing a monotone submodular function subject to $O(1)$ covering and one packing constraint that outputs for any $\varepsilon > 0$ in $n^{O(1/\varepsilon)}$ time a e-approximate solution with profit at least $\left(\frac{1}{2} - \varepsilon\right) P$ and with weight at most $(1+\varepsilon)W$.*

Open Problems

- Can we get a tight $1 - 1/e$ approximation for 1 cardinality and 1 covering constraint without any violation or the problem becomes harder?
- Can we get a $O(1)$ approximation for 1 covering and 1 packing constraint by violating only one constraint?
- Which other problems can benefit from our greedy DP technique?

Thank you