Near-Optimal Distributed Shortest Paths and Transshipment

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Shortest Transshipment Problem (STP)

Given:

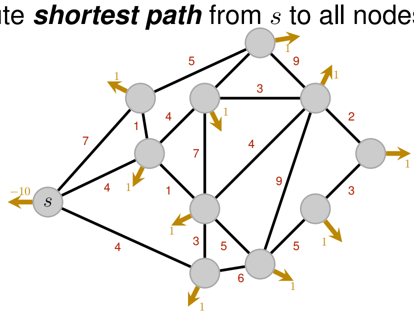
- undirected graph $G = (V, A)$, $n = |V|$
- edge weights $w \in \mathbb{N}^m$
- node demands $b \in \mathbb{Z}^n$

Goal: ***Ship goods*** from negative demand nodes to positive demand nodes, minimizing the cost.

$$\min\{\|Wx\|_1 : Ax = b\} = \max\{b^T y : \|W^{-1}A^T y\|_\infty \leq 1\}$$

Single-Source-Shortest-Path (SSSP)

Compute ***shortest path*** from s to all nodes (SSSP).



SSSP is the special case of STP with $b = \mathbb{1} - n\mathbb{1}_s$.

Gradient Descent Approach

Idea: Solve differentiable variant of Reciprocal LP

$$\min\{\|W^{-1}A^T\pi\|_\infty : b^T\pi = 1\}$$

using a ***potential function*** $\Phi_\beta(\pi) := \mathrm{lse}_\beta(W^{-1}A^T\pi)$.

Algorithm: `grad_desc`

$\pi \leftarrow \alpha$-approx., β s.t. $\Phi_\beta(\pi) \in [\frac{4\ln(2m)}{\varepsilon\beta}, \frac{5\ln(2m)}{\varepsilon\beta}]$

repeat

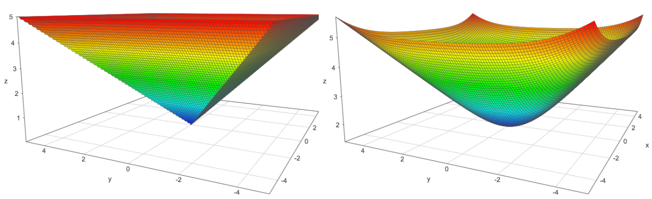
- $\tilde{b} \leftarrow P^T \nabla\Phi_\beta(\pi)$, where $P \leftarrow I - \pi b^T$
- $\tilde{h} \leftarrow \alpha$-approx. of $\max\{\tilde{b}^T h : \|W^{-1}A^T h\|_\infty \leq 1\}$
- $\delta \leftarrow \frac{\tilde{b}^T\tilde{h}}{\|W^{-1}A^T P\tilde{h}\|_\infty}$
- **if** $\delta > \frac{\varepsilon}{8\alpha}$ **then** $\pi \leftarrow \pi - \frac{\delta}{2\beta\|W^{-1}A^T P\tilde{h}\|_\infty} P\tilde{h}$.

until $\delta \leq \frac{\varepsilon}{8\alpha}$

Here $\mathrm{lse}_\beta(x)$ is the so-called log-sum-exp function

$$\mathrm{lse}_\beta(x) := \frac{1}{\beta}\ln\left(\textstyle\sum_{i\in[d]} \exp(\beta x_i) + \exp(-\beta x_i)\right)$$

that approx. the ∞-norm. A 2-dim. example, for $\beta = 1$:



Results

Theorem 1: Given α-oracle for STP, one can compute solutions x, y for STP s.t. $\|Wx\|_1 \leq (1+\varepsilon)b^T y$ with $\tilde{O}(\varepsilon^{-3}\alpha^2)$ oracle calls.

Theorem 2: Can compute solution y for SSSP s.t. $\frac{y_v^*}{1+\varepsilon} \leq y_v \leq y_v^*$ for each v with polylog$(n, \|w\|_\infty)$ calls to `grad_desc`.

Implementing this framework with an oracle that computes an **optimal solution on an α-spanner** of G yields the following results.

Results in Distributed/Streaming Models:

1. ***Broadcast congest:*** $(1+\epsilon)$-approx. SSSP in $\tilde{O}(\epsilon^{-O(1)}(\sqrt{n} + D))$ rounds
2. ***Broadcast congested clique:*** $(1+\epsilon)$-approx. STP and SSSP in $\tilde{O}(\epsilon^{-O(1)})$ rounds.
3. ***Multipass streaming:*** $(1+\epsilon)$-approx. STP and SSSP in $\tilde{O}(n)$ space and $\tilde{O}(\epsilon^{-O(1)})$ passes.

*Note that the upper bound for the **Broadcast Congest model** is the first to **match the well-known lower bound** of $\Omega(\sqrt{n}/\log n + D)$ up to logarithmic factors.*

References

[BKKL16] Ruben Becker, Andreas Karrenbauer, Sebastian Krinninger, and Christoph Lenzen. *Near-Optimal Approximate Shortest Paths and Transshipment in Distributed and Streaming Models.* 2016.

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