Near-Optimal Approximate Shortest Paths and Transshipment in Distributed and Streaming Models

Ruben Becker, Andreas Karrenbauer, Sebastian Krinninger and Christoph Lenzen

Max Planck Institute for Informatics

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Given **undirected** graph $G = (V, E)$
- **edge weights** $w \in \mathbb{N}^m$,
- **node demands** $b \in \mathbb{Z}^n$ s.t. $1^T b = 0$

**Undirected Transshipment**

$$
\min \{ \|Wx\|_1 : Ax = b\} = \max \{b^Ty : \|W^{-1}A^Ty\|_\infty \leq 1\}.
$$

*Single Source Shortest Path is special case $b = 1 - n1_s$.*

**Lemma**

The problem $\min \{ \|W^{-1}A^T\pi\|_\infty : b^T\pi = 1\}$ is equivalent.

**Approximately solve by**

minimizing $\Phi_\beta(\pi) := \text{lse}_\beta(W^{-1}A^T\pi)$, where $\text{lse}_\beta(v)$ is log-sum-exp.
Transshipment & Single Source Shortest Path

Given an undirected graph $G = (V, E)$ with edge weights $w \in \mathbb{N}$, node demands $b \in \mathbb{Z}$ s.t. $\sum b = 0$.  

Undirected Transshipment

$$\min \{ \|Wx\|_1 : Ax = b \} = \max \{ b^T y : \|W^{-1}A^T \pi\|_\infty \leq 1 \}.$$  

Single Source Shortest Path is special case $b = 1$.  

Lemma

The problem $\min \{ \|W^{-1}A^T \pi\|_\infty : b^T \pi = 1 \}$ is equivalent.  

Approximately solve by

minimizing $\Phi_\beta(\pi) := \text{lse}_\beta(W^{-1}A^T \pi)$, where $\text{lse}_\beta(v)$ is log-sum-exp.
Approach: Gradient Descent on $\Phi_\beta(\pi)$

Potential Function Change on Update

$$\Phi_\beta(\pi) - \Phi_\beta(\pi - h) \geq \nabla \Phi_\beta(\pi)^T h - \beta \| W^{-1} A^T h \|_\infty^2$$

- This suggests to compute $h$ by solving
  $$\max \{ \nabla \Phi_\beta(\pi)^T h : \| W^{-1} A^T h \|_\infty \leq 1 \}.$$

- Another transshipment problem with demand vector $\nabla \Phi_\beta(\pi)$.
- Using oracle for $\alpha$-approximation $h$ yields:

Theorem

Using oracle for $\alpha$-approximate transshipment, one can compute solutions $x, y$ s.t. $\| W x \|_1 \leq (1 + \varepsilon) b^T y$ with $\tilde{O}(\varepsilon^{-3} \alpha^2)$ oracle calls.

(Truth a bit more complicated: The feasible direction $h$ must satisfy $b^T h = 0$.)
## Results: Distributed/Streaming Models

### $\alpha$-oracle

Use the optimal solution on sparse $\alpha$-spanner.

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<thead>
<tr>
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<th>Previous</th>
<th>New</th>
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<tbody>
<tr>
<td>Brdc Congest</td>
<td>$\varepsilon^{-o(1)}(n^{1/2} + o(1) + D^{1+o(1)})$ rounds</td>
<td>$\tilde{O}(\varepsilon^{-O(1)}(n^{1/2} + D))$ rounds</td>
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<tr>
<td>SSSP</td>
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<tr>
<td>Brdc Congest Clique</td>
<td>$\varepsilon^{-o(1)} n^{o(1)}$ rounds</td>
<td>$\tilde{O}(\varepsilon^{-O(1)})$ rounds</td>
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<tr>
<td>SSSP/STP</td>
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<td>Multipass streaming</td>
<td>$\varepsilon^{-o(1)} n^{o(1)}$ passes</td>
<td>$\tilde{O}(\varepsilon^{-O(1)})$ passes</td>
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<td>SSSP/STP</td>
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- Matches lower bounds in terms of $n$ up to log-factors.
- Provides hope towards parallel solutions of SSSP/STP?

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Thank you for your attention!