Maximizing a Monotone Submodular Function Subject to a Covering and a Packing Constraint

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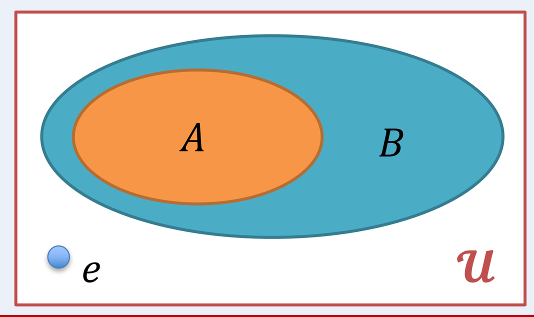
Submodular Function

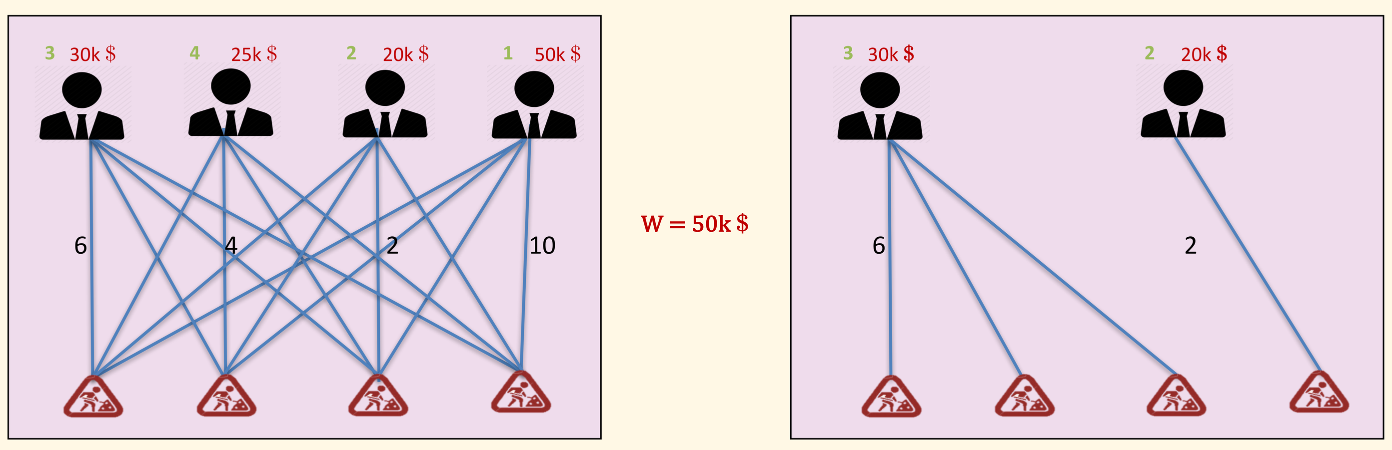
A set function $f: 2^{\mathcal{U}} \rightarrow \mathbb{R}^+$ on all subsets of a ground set $\mathcal{U}$ such that $\forall A \subseteq B \subseteq \mathcal{U}, e \in \mathcal{U} \backslash B$,

$$f(A \cup \{e\}) - f(A) \geq f(B \cup \{e\}) - f(B)$$



Monotone: $f(B) \geq f(A)$

Motivation & Problem Definition



We are given a profit function $p: \mathcal{U} \rightarrow \mathbb{N}$, a weight function $w: \mathcal{U} \rightarrow \mathbb{N}$, a profit requirement P and a budget B.

Goal: *max* $f(A)$ over all $A \subseteq \mathcal{U}$ *such that* $p(A) \geq P$ and $w(A) \leq W$.

Feasibility NP-Hard (Subset-Sum Problem)

State of the Art

- *One Cardinality Constraint*
  - Greedy $(1 - 1/e)$ – apx [Nemhauser et al. , Math. Program.'78]
  - Hardness: $(1 - 1/e)$ [Nemhauser et al. , Math. O.R., 1978], [Feige, STOC'96]
- *One Packing Constraint*
  - Greedy $(1 - 1/e)$ – apx [Sviridenko, O.R. Lett.'04]
- *$O(1)$ Packing Constraints*
  - Multilinear relaxation: $(1 - 1/e - \epsilon)$ – apx [Kulik et al. , SODA'09]
- *Matroid Constraints*
  - Multilinear relaxation: 0.309 – apx [Vondrák, STOC'08], [Calinescu et al., IPCO'07]

Greedy Dynamic Program

- Outputs an approximate solution which is best of all possible greedy chains.

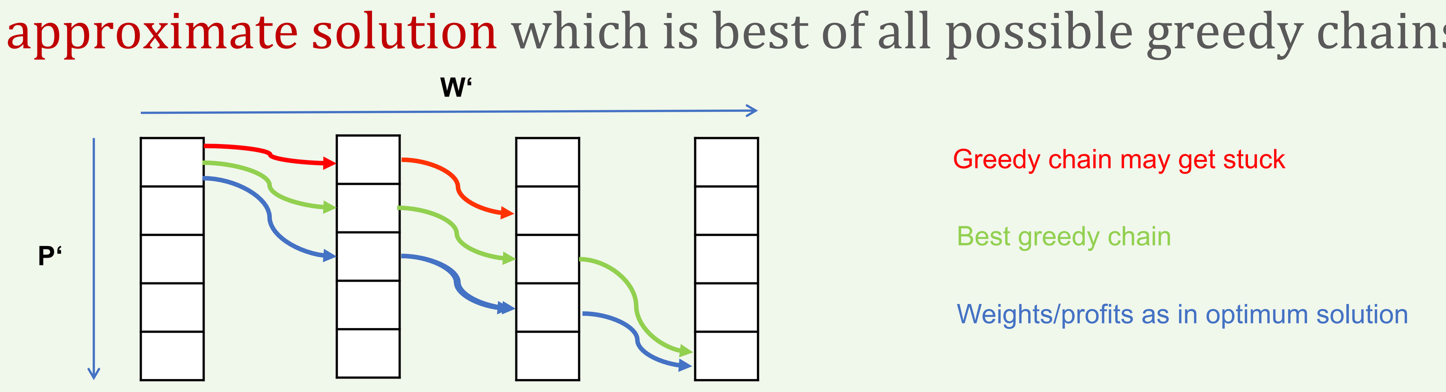


Table entry: $T[l, P', W']$ stores l-element approximate solution with profit P' and weight W'

To compute $T[l, P', W']$ pick best way to extend an entry $T[l-1, P'', W'']$ by one element in a feasible manner

Output: Best entry $T[l, P', W']$ with $P' \geq \frac{P}{2}$ and $W' \leq W$.

Forbidden Sets

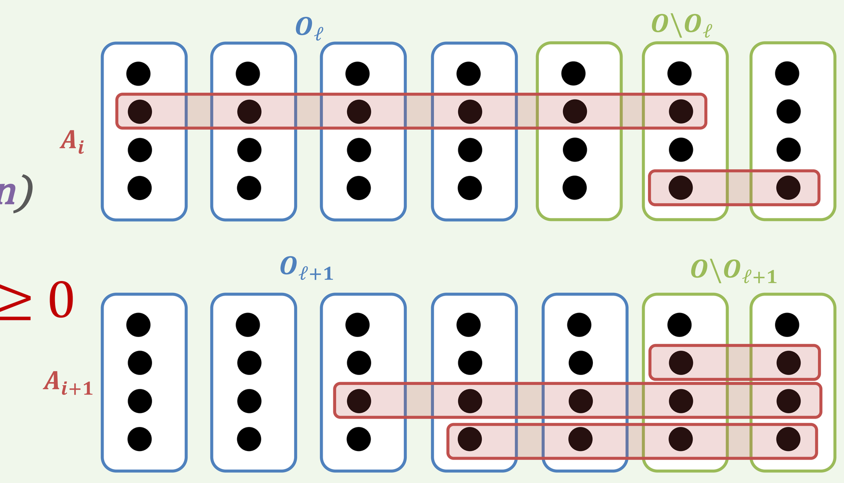
Crucial to reduce the profit violation from $2 + \varepsilon$ down to $1 + \varepsilon$.

Works for one covering and one packing constraint.

Idea: For each $T[P', W']$, forbid the suffix set $F_{W'}$ of ordered set by non-increasing $\frac{p(e)}{w(e)}$ such that $p(T[P', W'] \cup F_{W'}) \geq P$ and $w(T[P', W'] \cup F_{W'}) \leq W$.

Analysis

- $O = \{e^1, e^2, \ldots, e^k\}$. Let $O^i = \{e^1, e^2, \ldots, e^i\}$ and $g(e^i) = f(O^i) - f(O^{i-1})$ *(Incremental gain function)*



- **Lemma:** There is a $0 \leq W' \leq W$ and a $P' \geq 0$ with $P' + p(F_{W'}) \geq P$ such that

$$f(T[P', W'] \cup F_{W'}) \geq \frac{1}{4} f(O)$$

Proof idea:

Keep adding elements from $O \backslash F_{W'}$ inductively to current solution $T[P', W']$ and to the l-subset $O_l = \{e_1, \ldots, e_l\} \subseteq O$ until $f(T[P', W'] \cup \{e\}) - f(T[P', W']) \geq \frac{1}{2} g(e)$. If no such element exists, then, $f(T[P', W']) \geq \frac{1}{2} g(O \backslash (F_{W'} \cup O_l))$ and $f(T[P', W']) \geq \frac{1}{2} g(O_l)$.

Repeating the same reasoning with elements in $O \cap F_{W'}$ proves the lemma.

Our Results

Theorem 1. *There is an algorithm for maximizing a monotone submodular function subject to one covering and one packing constraint that outputs for any $\varepsilon > 0$ in $n^{O(1/\varepsilon)}$ time a **4** - approximate solution with profit at least $(1 - \varepsilon)P$ and with weight at most $(1 + \varepsilon)W$.*

Corollary. *There is a **2.6** - approximation algorithm for k-median problem with non-uniform and hard capacities if the underlying metric space has only two possible distances.*

Factor-revealing LP

$\min\ a_n$ *subject to* (LP)	$\max \sum_{i=1}^{n} \frac{i}{n} y_i$ *subject to* (DUAL)
$a_1 \geq \left(1 - \frac{1}{n}\right) o_1;$	$x_i + y_i - x_{i+1} \leq 0 \quad \forall i \in [n-1];$
$a_i \geq a_{i-1} + \left(1 - \frac{i}{n}\right) o_i \quad \forall i \in [n] \backslash \{1\};$	$x_n + y_n \leq 1;$
$a_i \geq \frac{i}{n}\left(1 - \sum_{j=1}^{i} o_j\right) \quad \forall i \in [n];$	$\sum_{j=i}^{n} \frac{j}{n} y_j - \left(1 - \frac{i}{n}\right) x_i \leq 0 \quad \forall i \in [n];$
$a_i \geq 0,\ o_i \geq 0 \quad \forall i \in [n].$	$x_i \geq 0,\ y_i \geq 0 \quad \forall i \in [n].$

Some more results

Theorem 2. *There is an algorithm for maximizing a monotone submodular function subject to one covering and one packing constraint that outputs for any $\varepsilon > 0$ in $n^{O(1/\varepsilon)}$ time a **e** - approximate solution with profit at least $\left(\frac{1}{2} - \varepsilon\right) P$ and with weight at most $(1 + \varepsilon)W$.*

Future Directions

- Extend the factor-revealing LP analysis to get ***e**-approximate* solution for one cardinality constraint and one covering constraint.
- Apply our approach to other settings where the greedy works

Conclusion: We give the first greedy DP based approximation algorithm that has a potential to handle complex constraints.

References

[1] Joachim Spoerhase and Sumedha Uniyal. "Maximizing a Monotone Submodular Function Subject to a Covering and a Packing Constraint" *Ongoing work*, *https://users.aalto.fi/~uniyals1/resources/MixSubmodular-17.pdf*.

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